A SOLUTION OF THE VOLterra EQUATION

\[ f(x) = g(x) + \int_0^x k(x,t)f(t)dt \]

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Abstract: In this paper we introduce a series solution for Volterra Equation with limits of integration from 0 to \( x^n \) which is a generalization of the classical Volterra equation for \( n = 1 \).

Keywords: integral equation, Maclaurin series, Leibniz Rule, Functional-differential equations, Volterra equation

I. Introduction

The Volterra integral equation with limits of integral from 0 to \( x^n \), where \( n \) is a positive integer is of the form

\[ f(x) = g(x) + \int_0^x k(x,t)f(t)dt \]

where \( g \) and \( k \) are known functions, and \( f \) is the unknown function to be found. As far as the authors know, no solution for the case of \( n > 1 \) is given. Actually for \( n = 1 \), the equation leads to a differential equation. While for \( n > 1 \) the equation leads to a functional differential equation. We assume here that the functions \( g \), \( f \), and \( k \) are analytic at 0. So, we can use the Maclaurian series technique to get our solution, namely

\[ f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + ..., \]

So, our process simply, is to determine the derivatives at 0 of \( f(x) \) in terms of the derivatives at 0 of \( g(x) \) and \( k(x) \) to get the solution of

\[ f(x) = g(x) + \int_0^x k(x,t)f(t)dt, \quad n > 1, \]

II Solution of the equation

\[ f(x) = g(x) + \int_0^x k(x,t)f(t)dt \]

By Leibniz rule find \( f'(x), f''(x), f'''(x), \ldots \)

\[ f'(x) = g'(x) + \int_0^x \frac{\partial}{\partial x} k(x,t)f(t)dt + (nx^{n-1})k(x,x^n)f(x^n) \]

\[ = g'(x) + \int_0^x \frac{\partial}{\partial x} k(x,t)f(t)dt + R_1(x), \]

where

\[ R_1(x) = (nx^{n-1})k(x,x^n)f(x^n), \]

\[ R_1(0) = 0 \quad \text{if} \quad n > 1 \]
\[ f''(x) = g''(x) + \int \frac{\partial^2}{\partial x^2} k(x,t) f(t) dt + [(2n x^{n-1}) k'(x,x^n) \\
+ n(n-1)x^{n-2} k(x,x^n)] f(x^n) + (n^2 x^{2n-2}) k(x,x^n) f'(x^n) \]

\[ = g''(x) + \int \frac{\partial^2}{\partial x^2} k(x,t) f(t) dt + R_2(x) \]

Where

\[ R_2(x) = [(2n x^{n-1}) k'(x,x^n) + n(n-1)x^{n-2} k(x,x^n)] f(x^n) \\
+ (n^2 x^{2n-2}) k(x,x^n) f'(x^n) \]

\[ R_2(0) = 0 \quad \text{if} \quad n > 2 \]

\[ f'''(x) = g'''(x) + \int \frac{\partial^3}{\partial x^3} k(x,t) f(t) dt + \frac{n(n-1)(n-2)x^{n-3}k(x,x^n)}{6} + \frac{3(n^2-1)x^{2n-2}k'(x,x^n)}{4} \]

\[ + \frac{(3n^2(n-1)x^{2n-3}k(x,x^n))}{6} + \frac{3n^2(n-1)x^{3n-3}k(x,x^n)}{6} f'(x^n) + \frac{3n^2(n-1)x^{3n-3}k(x,x^n)}{6} f''(x^n) \]

\[ f'''(x) = g'''(x) + \int \frac{\partial^3}{\partial x^3} k(x,t) f(t) dt + R_2(x), \]

Where

\[ R_3(x) = \frac{n(n-1)(n-2)x^{n-3}k(x,x^n)}{6} + \frac{3n^2(n-1)x^{2n-2}k'(x,x^n)}{4} \]

\[ + \frac{(3n^2(n-1)x^{2n-3}k(x,x^n))}{6} + \frac{3n^2(n-1)x^{3n-3}k(x,x^n)}{6} f'(x^n) + \frac{3n^2(n-1)x^{3n-3}k(x,x^n)}{6} f''(x^n) \]

\[ R_3(0) = 0 \quad \text{if} \quad n > 3 \]

\[ f^{(4)}(x) = g^{(4)}(x) + \int \frac{\partial^4}{\partial x^4} k(x,t) f(t) dt + \frac{n(n-1)(n-2)(n-3)x^{n-4}k(x,x^n)}{24} \]

\[ + \frac{(4n(n-1)(n-2)x^{n-3}k'(x,x^n)}{4} + 6n(n-1)x^{n-2}k''(x,x^n) + \frac{4n^2(n-1)x^{n-3}k'''(x,x^n)}{4} f(x^n) \]

\[ + \frac{[(n^2(n-1)(7n-11)x^{2n-4}k(x,x^n)}{4} + \frac{(12n^2(n-1)x^{2n-3}k'(x,x^n)}{4} + \frac{(6n^2(n-1)x^{2n-3}k''(x,x^n)}{4} f'(x^n) + \frac{[(6n^2(n-1)x^{3n-4}k(x,x^n)}{4} + \frac{4n^3x^{3n-3}k'(x,x^n)}{4} f''(x^n) + \frac{n^4x^{4n-3}k(x,x^n)}{4} f'''(x^n) \]

\[ R_4(0) = 0 \quad \text{if} \quad n > 4 \]
\[ f^{(5)}(x) = g^{(5)}(x) + \int_0^x \frac{\partial^5}{\partial x^5} k(x,t)f(t) \, dt \]
\[ + [(n(n-1)(n-2)(n-3)(n-4)x^{n-5}k(x,x')]

The general form
\[ f^{(m)}(x) = g^{(m)}(x) + \int_0^x \frac{\partial^m}{\partial x^m} k(x,t)f(t) \, dt + R_m(x), \]

where,
\[ R_m(x) = \sum_{j=1}^{m} \sum_{c=1}^{j} a_{mjc} p(n, j-c+1)x^{(m-j+1)n-(m-c+1)k(c-1)(x,x^n)} f^{(m-j)}(x^n), \]

where
\[ a_{mjc} \text{ is real number and } p(n, j-c+1) = n(n-1)(n-2)...(n-j+c) \]
\[ R_m(0) = 0, \text{ when } n > m. \text{ thus, replacing } x \text{ by } 0 \text{ we obtain} \]
\[ f(0) = g(0) \]
\[ f'(0) = \begin{cases} g'(0) + R_1(0), & n = 1 \\ g'(0), & n > 1 \end{cases} \]
\[ f''(0) = \begin{cases} g''(0) + R_2(0), & n = 2 \\ g''(0), & n > 2 \end{cases} \]
\[ f'''(0) = \begin{cases} g'''(0) + R_3(0), & n = 3 \\ g'''(0), & n > 3 \end{cases} \]
\[ f^{(4)}(0) = \begin{cases} g^{(4)}(0) + R_4(0), & n = 4 \\ g^{(4)}(0), & n > 4 \end{cases} \]

The general form at \( x = 0 \) is
\[ f^{(m)}(0) = \begin{cases} g^{(m)}(0) + R_m(0), & n = m \\ g^{(m)}(0), & n > m \end{cases} \]

Applying Maclaurin series
\[ f(x) = f(0) + f'(0) \frac{x}{1!} + f''(0) \frac{x^2}{2!} + f'''(0) \frac{x^3}{3!} + ..., \]

So the solution...
\[ f(x) = \sum_{m=1}^{\infty} \left( g^{(m)}(0) + R_m(0) \right) \frac{x^m}{m!} \]  
\[ = g(x) + \sum_{m=1}^{\infty} \left( R_m(0) \right) \frac{x^m}{m!} \]  (6)  
(7)  

The solution is divided into three types at \( n = 1, n = 2 \), and \( n > 2 \). When \( n = 1 \) the solution is in the form

\[ f(x) = g(x) + \sum_{m=1}^{\infty} \left[ \sum_{j=1}^{m} a_{mj} k^{(m-j)}(0,0) f^{(j-1)}(0) \right] \frac{x^m}{m!}, \]  (8)  

where \( a_{mj} \) is a real number

which is the known solution of Volterra equation (\( n = 1 \)).

When \( n = 2 \) the solution is in the form

\[ f(x) = g(x) + \sum_{j=2}^{\infty} \left[ k^{(j-2)}(0,0) f(0) \right] \frac{x^j}{(j-2)!} + \sum_{m=4}^{\infty} \left[ a_{m} k^{(m-4)}(0,0) f''(0) \right] \frac{x^m}{m!}, \]  (9)  

where \( a_m \) is a real number

When \( n > 2 \) the solution is in the form

\[ f(x) = g(x) + g(0) \sum_{j=n}^{\infty} \left[ k^{(j-n)}(0,0) \right] \frac{x^j}{(j-n)!} \]  (10)  

III. Examples

(1) The solution of the integral equation

\[ f(x) = g(x) + \int_{0}^{x} k(x,t) f(t) dt, \]  

\[ f(x) = g(x) + g(0) \sum_{j=3}^{\infty} \left[ k^{(j-3)}(0,0) \right] \frac{x^j}{(j-3)!} \]  

(2) The solution of the integral equation

\[ f(x) = 1 - x^3 + \int_{0}^{x} (1 - 2t^3 + x^6) f(t) dt, \]  

\[ f(x) = 1 \]  

References

