Abstract— In this work, we introduce the Fock space $F_v(C)$ associated to the Airy operator, and we establish Heisenberg-type uncertainty principle for this space. Next, we give an application of the theory of extremal function and reproducing kernel of Hilbert space, to establish the extremal function associated to a bounded linear operator $T : F_v(C) \rightarrow H$, where $H$ be a Hilbert space. Finally, we come up with some results regarding the extremal functions, when $T$ is the difference operator and the Dunkl-difference operator, respectively.

Keywords—component, Airy-type Fock space, Tikhonov regularization, Heisenberg-type uncertainty principle.

I. Introduction

The study of several generalizations of the classical Fock spaces has a long and rich history in many different settings (see for instance [11], [12]). In this work, we will try to generalize Airy-type Fock space, to establish Heisenberg-type uncertainty principle for this space; and to give an application of the theory of reproducing kernels to the Tikhonov regularization on this generalized Fock space.

The generalized Airy operator (or hyper-Bessel operator [14, 15]) is third singular differential operator given by

$$L_v = \frac{d^3}{dx^3} + \frac{3v}{x} \frac{d}{dx} - \frac{3v}{x^2} \frac{d}{dx},$$

where $v$ is a nonnegative real number. When $v = 0$, this operator becomes the third derivative operator for which some analysis were studied by Widder [18] and for some special value of $v$ the operator $L_v$ appeared as a radial part of the generalized Airy equation of a nonlinear diffusion type partial differential equation in $R^3$. Recently, in a nice and long paper, Cholewinski and Reneke [2] studied and extended, for the operator $L_v$, the well known theory related to some singular differential operator of second order for which the literature is extensive.

Next, Fitouhi et al. [7, 8] established a harmonic analysis related to this operator for examples the eigenfunctions, the generalized translation, the Fourier-Airy transform, the heat equation, the heat polynomials, the transmutation operators, …

During the last years, the Airy operator have gained considerable interest in various field of mathematics [3-6], [13-15] and in certain parts of quantum mechanics [1]. The results of this work will be useful when discussing the Fock space associated to this operator.

This space is the background of some applications in this contribution. Especially,

- we study the Airy operator and its adjoint operator on the Airy-type Fock space.
- We establish Heisenberg-type uncertainty principle for the Airy-type Fock space.
- We give an application of the theory of extremal function and reproducing kernel of Hilbert space, to establish the extremal function associated to a bounded linear operator $T : F_v(C) \rightarrow H$, where $H$ be a Hilbert space.

- We come up with some results regarding the extremal functions associated to the following operators:
  - The difference operator $Tf(z) := \frac{1}{z^3} (f(z) - f(0)).$
  - And the Dunkl-difference operator $Tf(z) := \frac{1}{2z^3} (f(z) - f(-z))$.

The contents of the paper are as follows. In Section 2, we introduce the Airy-type Fock space $F_v(C)$. In Section 3, we establish Heisenberg-type uncertainty principle for this space. In Section 4, we give an application of the theory of reproducing kernels to the Tikhonov regularization problem for bounded linear operator $T : F_v(C) \rightarrow H$, where $H$ be a Hilbert space. Next, we come up with some results regarding the Tikhonov regularization problem for the difference operator and the Dunkl-difference operator, respectively.

II. Airy-type Fock space

Let $z \in C$ and $\alpha_k = e^{2\pi i k - 1} / 3$, $k = 1, 2, 3$. A function $u(z)$ is called 3-even if $u(\alpha_k z) = u(z)$.

For $\lambda \in C$, the initial problem $L_4 u(z) = \lambda^3 u(z)$, with $u(0) = 1$ and $u^{(k)}(0) = 0$, $k = 1, 2$ admits a unique analytic solution on $C$, which will be denoted by $G_4(\lambda z)$ and expanded in power series as

$$G_4(\lambda z) = \sum_{n=0}^{\infty} \alpha_n(\lambda) z^n,$$

where $\alpha_n(\lambda) := \frac{1}{n!} \left( \frac{\partial^n}{\partial \lambda^n} G_4(\lambda z) \right)_{z=0}$.
The function $G_v(\lambda z)$ is 3-even and defined as the hypergeometric function (see [2]),

$$G_v(\lambda z) = F_v \left[ 1, v + \frac{2}{3} (\frac{\lambda}{3})^3 \right].$$

In particular $|G_v(\lambda z)| \leq e^{||f||^2}$ and $G_v(\lambda z) = \cos_v(-z) = \sum_{n=0}^{\infty} (\lambda z)^{3n} (3n)!$.

We define the Airy-type Fock space $F_v(C)$ as the prehilbertian space of 3-even entire functions $f(z) = \sum_{n=0}^{\infty} a_n z^{3n}$ on $C$ such that

$$\|f\|_{F_v(C)}^2 = \sum_{n=0}^{\infty} |a_n|^2 \alpha_{3n}(v) < \infty.$$ 

Let $f$ and $g$ be in $F_v(C)$ such that $f(z) = \sum_{n=0}^{\infty} a_n z^{3n}$ and $g(z) = \sum_{n=0}^{\infty} b_n z^{3n}$. The inner product in $F_v(C)$ is given by

$$\langle f, g \rangle_{F_v(C)} := \sum_{n=0}^{\infty} a_n \bar{b_n} \alpha_{3n}(v).$$

The space $F_v(C)$ satisfies the following properties.

(i) The function $K_v$ given by $K_v(w, z) = G_v(wz)$ is a reproducing kernel for the space $F_v(C)$.

(ii) If $f \in F_v(C)$, then $|f(z)| \leq e^{\frac{z^3}{3}} \|f\|_{F_v(C)}$, $z \in C$.

(iii) The space $F_v(C)$ equipped with the inner product $\langle \cdot, \cdot \rangle_{F_v(C)}$ is a Hilbert space; and the set \[ \left\{ \frac{z^{3n}}{\sqrt{\alpha_{3n}(v)}} \right\}_{n \in \mathbb{N}} \] forms an Hilbert’s basis for the space $F_v(C)$.

### III- Heisenberg-type uncertainty principle

We consider the following lemmas.

**Lemma 3.1.** Let $L_v : F_v(C) \rightarrow F_v(C)$, then its adjoint operator $L_v^*$ is given by $L_v^* f(z) = z^3 f(z)$.

**Lemma 3.2.** $[L_v, L_v^*] := L_v L_v^* - L_v^* L_v = (6+9v)I + B_v$, where

$$B_v f(z) = 9z^2 \frac{d^2}{dz^2} f(z) + 18(1+v)z \frac{d}{dz} f(z).$$

**Lemma 3.3.** Let $F_v(C)$. Then

$$\|L_v f\|^2_{F_v(C)} = \|L_v^* f\|^2_{F_v(C)} + (6+9v) \|f\|^2_{F_v(C)} + \langle f, B_v f \rangle_{F_v(C)}.$$ 

By applying Lemma 3.1, Lemma 3.2, Lemma 3.3 and the following theorem.

**Theorem 3.4.** (See [9, 10]). Let $A$ and $B$ be self adjoint operators on a Hilbert space $H$. Then we have

$$\|A - a\|_{H} \|B - b\|_{H} \geq \frac{1}{2} \|[[A, B], f, f]_{H}\|,$$

for all $f \in \text{Dom}(AB) \cap \text{Dom}(BA)$, and all $a, b \in R$.

We obtain the following Heisenberg-type uncertainty principle for the Airy-type Fock space $F_v(C)$.

**Theorem 3.5.** Let $f \in F_v(C)$. For all $a, b \in R$, we have

$$\|L_v + z^3 - a\|_{F_v(C)} \|L_v - z^3 + ib\|_{F_v(C)} \geq A_v f,$$

where $A_v f = (6+9v) \|f\|^2_{F_v(C)} + \langle B_v f, f \rangle_{F_v(C)}$.

This uncertainty principle can be written as the following form.

**Theorem 3.6.** Let $f \in F_v(C)$. Then

$$\Delta_v^x (f) \Delta_v^y (f) \geq \|f\|^2_{F_v(C)} (A_v f)^2,$$

where

$$\Delta_v^x (f) = \|f\|^2_{F_v(C)} \|L_v \pm z^3\|_{F_v(C)}^2 - \langle (L_v \pm z^3), f \rangle_{F_v(C)}^2.$$ 

### IV- Extremal functions on $F_v(C)$

Let $\lambda > 0$ and let $T : F_v(C) \rightarrow H$ be a bounded linear operator from $F_v(C)$ into a Hilbert space $H$. We denote by $\langle f, g \rangle_{L,\lambda}$ the inner product defined on the space $F_v(C)$ by

$$\langle f, g \rangle_{L,\lambda} := \lambda \langle f, g \rangle_{F_v(C)} + \langle Tf, Tg \rangle_H.$$ 

By using the theory reproducing kernels of Hilbert space and building the ideas of Saitoh [16,17] we establish the extremal function associated to the operator $T$ on the Airy-type Fock space $F_v(C)$. 

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Theorem 4.1. Let $\lambda > 0$. The Fock space $\left( F_{\lambda}(C), \langle \cdot, \cdot \rangle_{F_{\lambda}} \right)$ possesses a reproducing kernel $K_{T,\lambda}(w, z) ; w, z \in C$, which satisfies the equation 

$$(\lambda I + T^*T)K_{T,\lambda}(w, \cdot) = K_{\lambda}(w, \cdot).$$

Moreover the kernel $K_{T,\lambda}(w, z)$ satisfies the following properties.

(i) $\|K_{T,\lambda}(w, \cdot)\|_{F_{\lambda}(C)} \leq \frac{e}{2\lambda}$.

(ii) $\|TK_{T,\lambda}(w, \cdot)\|_{H} \leq \frac{e}{2\lambda}$.

(iii) $\|T^*TK_{T,\lambda}(w, \cdot)\|_{F_{\lambda}(C)} \leq \frac{e}{\lambda^2}$.

From this theorem we obtain the main result of this section.

Theorem 4.2. For any $h \in H$ and for any $\lambda > 0$, there exists a unique function $f_{\lambda, h}^*$, where the infimum

$$\inf_{f \in F_{\lambda}(C)} \left( \|f\|_{F_{\lambda}(C)}^2 + \|h - Tf\|_H^2 \right)$$

is attained.

Moreover, the extremal function $f_{\lambda, h}^*$ is given by

$$f_{\lambda, h}^*(w) = \langle h, LK_{T,\lambda}(w, \cdot) \rangle_H$$

and satisfies the inequality

$$\|f_{\lambda, h}^*(w)\|_{H} \leq \frac{\|h\|_{H}^2}{2\lambda^2}.$$ 

Application 4.3. Let $H$ be the prehilbertian space of $3$-even entire functions $f(z) = \sum_{n=0}^{\infty} a_n z^{3n}$ on $C$ such that

$$\|f\|_{H}^2 = \sum_{n=0}^{\infty} |a_n|^2 \alpha_{3n+3} = \infty.$$ 

The space $H$ is a Hilbert space with Hilbert’s basis $\left\{ \frac{z^n}{\alpha_{3n+3}(v)} \right\}_{n \in \mathbb{N}}$ and reproducing kernel $S_h(w, z) = \sum_{n=0}^{\infty} \frac{z^n}{\alpha_{3n+3}(v)}$.

(i) Let $\bar{T}$ be the difference operator defined on $F_{\lambda}(C)$ by

$$\bar{T}f(z) := \frac{1}{z^2}(f(z) - f(0)).$$

The operator $\bar{T}$ maps continuously from $F_{\lambda}(C)$ into $H$, and we have

$$K_{T,\lambda}(w, z) = \frac{1}{\lambda} + \frac{1}{\lambda + 1} \left( G_{w}(z) - 1 \right),$$

and $f_{\lambda, h}^*(w) = \frac{1}{2\lambda^2} w^3 h(w)$.

(ii) Let $T$ be the Dunkl-difference operator defined on $F_{\lambda}(C)$ by $Tf(z) := \frac{1}{2z^3}(f(z) - f(-z))$.

The operator $T$ maps continuously from $F_{\lambda}(C)$ into $H$, and we have

$$K_{T,\lambda}(w, z) = \frac{1}{\lambda + 1} \sum_{n=0}^{\infty} \frac{z^n}{\alpha_{3n+3}(v)} + \frac{1}{\lambda + 2} \sum_{n=0}^{\infty} \frac{z^{n+3}}{\alpha_{3n+3}(v)},$$

and $f_{\lambda, h}^*(w) = \frac{1}{2(\lambda + 1)} w^3 (h(w) + h(-w))$.

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References


